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LETTER TO THE EDITOR

Zero-range potential model of a protruding stiffener

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Abstract. The point model of a protruding stiffener in a rigid screen is suggested. This model is formulated as the zero-range potential for the harmonic operator in Hilbert space and corresponds to a passive dipole source.

1. Introduction

The simple models of small obstacles that reproduce the main characteristics of the scattered fields are widely used in structural mechanics, acoustics and engineering. The main advantage of such models is in the existence of an explicit solution that can be expressed in the form of integrals or infinite series which makes the analysis the most simple one.

The explicit solution is possible if the homogeneity of the boundary value problem is violated in a separate point or in a finite or periodic set of points. Classical formulation of the 'boundary' conditions in a separate point is only possible if the order of the differential operator is high enough. The inhomogeneity can be introduced in the boundary condition if the generalized boundary conditions [1] are used. In this way all the models of point-wise defects in thin elastic plates are formulated [2–4].

If the order of the differential operator is not high enough, the value of the solution or its derivative in a separate point is, generally speaking, not defined and classical conditions in a point cannot be formulated. Indeed, the classical formulation of the boundary value problem deals with the weak solutions [5] that are from H^1 only, and there is no embedding form H^1 to the class of continuous functions.

In that case one needs to introduce some special technique that allows the condition in a point to be fixed. The most developed approach is based on the operator extensions theory and is referred as the zero-range potentials technique [6]. First zero-range potentials were introduced by Fermi in quantum mechanics and later became a popular tool in other sciences such as the acoustics of resonators with small openings [7], diffraction by narrow slits in acoustics [8] and electromagnetics, etc. The construction of the zero-range potential is performed in three steps [6, 9]. First the self-adjoint operator of the initial non-perturbed problem is restricted to functions that vanish near the chosen point that will become the potential centre. This yields the symmetric operator with some finite deficiency indices. In the second step the Neumann technique being in the restriction of the adjoint operator to the self-adjoint allows the class of all the zero-range potentials to be described. These zero-range potentials are formulated by means of 'boundary' conditions usually setting linear relations for the coefficients of the

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local asymptotics of the solution in a vicinity of the potential centre. The matrix of these linear relations parametrizes the zero-range potentials and is chosen at the third step of the procedure. In the technique of the zero-range potentials there is no tool for the choice of this matrix and some external considerations are used. Sometimes the matrix determining the boundary conditions is defined by the singular perturbation [10, 11].

The deficiency indices of the restricted harmonic operator in $\mathbb{R}^2_+$ are equal to $(1, 1)$ and the set of the zero-range potentials is one-parametric. All such zero-range potentials represent symmetric passive sources and the scattering amplitude is symmetric with the observation angle. The non-symmetric sources cannot be present in the problems if the deficiency indices are not increased. This can be achieved if instead of L_2, a wider in some sense space is used. These are spaces with indefinite metrics [12–14]. For the current state of the zero-range potential technique see [16].

Another approach is used in this paper. Here the model of the zero-range potential is constructed in a Hilbert space. We consider the problem of scattering by a protruding stiffener in a rigid screen and construct the zero-range potential with asymmetric scattered field that reproduces the scattering amplitude by the protruding stiffener in the low-frequency limit.

The approach is rather similar to that used in [15]. However, in [15] the operator is not formulated directly, but by means of Fourier transform. Besides, the parameter α that specifies the particular extension of the operator is involved in the formulation of the space itself. In this paper the parametrization is more natural.

2. Classical formulation and hints

Let the infinite rigid screen $y = 0$ have a protruding stiffener of height H. The boundary value problem that describes the diffraction by such a screen with a stiffener reads

$$\begin{aligned} &\Delta U(x, y) + k^2 U(x, y) = 0 \\ &\partial U/\partial y = 0 \qquad y = 0 \\ &\partial U/\partial x = 0 \qquad x = 0 \quad 0 < y < H. \end{aligned} \tag{1}$$

Here the time factor $\mathrm{e}^{-\mathrm{i}\omega t}$ is dropped and the field of acoustic pressure U is assumed to be independent of the z coordinate. That is, problem (1) is two-dimensional.

The wave process is generated by an incident plane wave

$$U^i(x, y) = \exp(\mathrm{i}kx\cos\varphi_0 - \mathrm{i}ky\sin\varphi_0)$$

which produces the reflected wave

$$U^r(x, y) = \exp(\mathrm{i}kx\cos\varphi_0 + \mathrm{i}ky\sin\varphi_0)$$

and the scattered field U^s which is to be found. The scattered field should satisfy the radiation condition at infinity and the total field $U = U^i + U^r + U^s$ is subject to the Meixner conditions in the points $(0, 0)$ and $(0, H)$.

The solution available to the author of the boundary value problem (1) belongs to Belinskiy (for a more complicated case of the elastic screen see [17]). The scattered field forms, at a large distance from the stiffener, the outgoing cylindrical wave

$$U^s \sim \sqrt{\frac{2\pi}{kr}}\mathrm{e}^{\mathrm{i}kr-\mathrm{i}\pi/4}\Psi(\varphi, \varphi_0) \qquad r \to +\infty \tag{2}$$

with the scattering pattern

$$\Psi(\varphi, \varphi_0) = \frac{\mathrm{i}}{2}(kH)^2\cos\varphi\cos\varphi_0\left\{1 + \frac{(kH)^2}{16}\left[1 - 4\log\left(\frac{kH}{4}\right)\right.\right.$$

$$-4C_E + \cos(2\varphi) + \cos(2\varphi_0)\Big] + \mathrm{O}((kH)^4 \log(kH))\Big\}. \tag{3}$$

Here C_E is the Euler constant.

The goal of this paper is to suggest the point model of the protruding stiffener that will reproduce the pattern $\Psi(\varphi, \varphi_0)$ in the principal order by kH. It is evident from (3) that the passive point source should have the directivity given by the cosine of the polar angle φ. That directivity corresponds to the point dipole source. Thus, formally, the Helmholtz equation in (1) gains the right-hand side

$$u_0 \delta'(x)\delta(y) \tag{4}$$

with some unknown amplitude u_0. This amplitude depends on the incident field in the vicinity of the origin and to get the factor $\cos\varphi_0$ one should take the derivative of the incident field. The rigorous formulation of the operator with a zero-range potential corresponding to the dipole source (4) is formulated below.

3. Zero-range potentials

3.1. The space

As it is noted above, the space L_2 does not allow the dipole sources to be introduced. Thus, another space should be used. This space should include the functions that solve the Helmholtz equation with the right-hand side given in (4). Besides, the functions from the domain of the harmonic operator in that space should be from C^1, that is their derivative should be defined at the origin.

Let us introduce the functions G_0 and G_1 that solve the equations

$$(-\Delta + \beta^2)G_0(x, y) = -\delta'(x)\delta(y)$$

and

$$(-\Delta + \alpha^2)G_1(x, y) = -G_0(x, y).$$

The local asymptotics of these functions are

$$G_0 \sim \frac{1}{\pi}\frac{x}{r^2} + \beta^2 \frac{x \log r}{2\pi} + \chi_0 x + \cdots \qquad r \to 0$$
$$G_1 \sim \frac{x \log r}{2\pi} + \chi_1 x + \cdots \qquad r \to 0$$

where

$$\chi_0 = \frac{\beta^2}{2\pi}(\log(\beta/2) + C_E - 1) \qquad \chi_1 = \frac{1}{2\pi}\left(\log(\beta/2) + C_E - \frac{1}{2}\right).$$

Let us associate with the function $\mathcal{U}(x, y)$ the pair $\mathcal{U} \equiv (U(x, y), u_0)$ by the following rule:

$$U(x, y) = \mathcal{U}(x, y) - u_0\, G_0(x, y) \in H^1(\mathbb{R}^2_+) \qquad u_0 \in \mathbb{C}. \tag{5}$$

One can easily check that for any field $\mathcal{U}(x, y)$ of the dipole source such a representation is possible.

The pairs $\mathcal{U}$ are considered as the elements of the space $\mathcal{H} = H^1(\mathbb{R}^2_+) + \mathbb{C}$ with the scalar product

$$(\mathcal{U}, \mathcal{V})_{\mathcal{H}} = (U, V) + \kappa_2(\nabla U, \nabla V) + \kappa_3 u_0 \overline{v_0} + \gamma_1\{(U, v_0 G_1) + (u_0 G_1, V)\}$$
$$+\gamma_2\{(\nabla U, v_0 \nabla G_1) + (u_0 \nabla G_1, \nabla V)\}. \tag{6}$$

Here the L_2 scalar product of the functions $U(x, y)$ and $V(x, y)$ is denoted by (U, V). The first three terms represent the scalar product in the space $H^1 \oplus \mathbb{C}$ and the last two terms introduce interaction (non-orthogonality) of the H^1 and $\mathbb{C}$ components. The positive constants $\kappa_2, \kappa_3, \gamma_1$ and γ_2 will be chosen later. However, there are some restrictions. The positive norm appears if κ_3 is large enough. Indeed the norm can be rewritten as

$$(\mathcal{U}, \mathcal{U})_{\mathcal{H}} = \|U + \gamma_1 u_0 G_1\|^2 + \kappa_2 \left\|\nabla U + \frac{\gamma_2}{\kappa_2} u_0 \nabla G_1\right\|^2 + \left(\kappa_3 - \gamma_1^2 \|G_1\|^2 - \frac{\gamma_2^2}{\kappa_2}\|\nabla G_1\|^2\right)|u_0|^2$$

and one concludes that

$$\kappa_3 = \kappa + \gamma_1^2 \|G_1\|^2 + \frac{\gamma_2^2}{\kappa_2}\|\nabla G_1\|^2 \qquad \kappa \geqslant 0. \tag{7}$$

Also, it is not difficult to check that the triangle rule is satisfied by the scalar product (6). Thus, the space $\mathcal{H}$ is a Hilbert one.

3.2. *Harmonic operator*

The harmonic operator $\mathcal{A}'$ in the space $\mathcal{H}$ is defined on the elements $\mathcal{U}$ with the functions $U(x, y)$ representable in the form

$$U(x, y) = u_1 G_1(x, y) + U_r(x, y) \tag{8}$$

where u_1 is arbitrary complex constant and U_r belongs to $H^3(\mathbb{R}^2_+)$ with Neumann boundary condition on $\{y = 0\}$. Then

$$\mathcal{A}'\mathcal{U} \equiv \begin{pmatrix} -\Delta U_r - \alpha^2 u_1 G_1 \\ -\beta^2 u_0 - u_1 \end{pmatrix} \tag{9}$$

belongs to $\mathcal{H}$. One can check that this formula appears if the harmonic operator is applied formally to the function $\mathcal{U}(x, y)$ and rule (5) is applied to the result.

The operator $\mathcal{A}'$ plays the role of the operator adjoint to the restricted one in the general scheme of [6]. It can be restricted to the self-adjoint operator $\mathcal{A}$. The singular coefficient u_0 should be dependent on the function $U(x, y)$. To find this dependence one considers the boundary form of the operator $\mathcal{A}'$

$$\begin{aligned}
(\mathcal{A}'\mathcal{U}, \mathcal{V})_{\mathcal{H}} - (\mathcal{U}, \mathcal{A}'\mathcal{V})_{\mathcal{H}} &= \frac{\gamma_2}{2}\left(\frac{\partial U_r(0,0)}{\partial x}\overline{v_0} - u_0 \overline{\frac{\partial V_r(0,0)}{\partial x}}\right) \\
&\quad + \frac{\kappa_2}{2}\left(\frac{\partial U_r(0,0)}{\partial x}\overline{v_1} - u_1 \overline{\frac{\partial V_r(0,0)}{\partial x}}\right) + W(u_1\overline{v_0} - u_0\overline{v_1}) \\
&\quad + (\gamma_1 - \gamma_2\alpha^2)[(U_r, v_0 G_0 + v_0(\alpha^2 - \beta^2)G_1 - v_1 G_1) \\
&\quad - (u_0 G_0 + u_0(\alpha^2 - \beta^2)G_1 - u_1 G_1, V_r)] \\
&\quad + (1 - \kappa_2\beta^2 + \gamma_2)[(U_r, v_1 G_0) - (u_1 G_0, V_r)]
\end{aligned}$$

where

$$W = \kappa_3 + (\alpha^2 - \beta^2)(\gamma_1\|G_1\|^2 + \gamma_2\|\nabla G_1\|^2).$$

To exclude integrals in the boundary form one lets

$$\gamma_1 = \gamma_2\alpha^2 \qquad \gamma_2 = \kappa_2\beta^2 - 1. \tag{10}$$

Finally, the restriction that makes the operator symmetric is formulated in the form of the linear relation

$$u_0 + \frac{\kappa_2}{\gamma_2} u_1 = \tilde{A}\left(\frac{\partial U_r(0,0)}{\partial x} + 2Wu_1\right) \tag{11}$$

with arbitrary real parameter $\tilde{A}$.

Theorem. *The operator $\mathcal{A}$ is self-adjoint in the space $\mathcal{H}$.*

Proof. From the above it is evident that the operator $\mathcal{A}$ is symmetric. Let us calculate the defficiency indices. We shall compute the defficiency index for a specially chosen real value of the spectral parameter and show that it is zero.

Consider the spectral problem

$$(A - \lambda)\mathcal{U} = \mathcal{F}$$

with negative λ. In component it reads

$$-(\Delta + \lambda)U_r - (\alpha^2 + \lambda)u_1 G_1 = F$$
$$-(\beta^2 + \lambda)u_0 - u_1 = f_0.$$

Besides, $\mathcal{U}$ should belong to the domain of $\mathcal{A}$, i.e. satisfying condition (11). By introducing the solutions U_f and U_g (due to $F, G_1 \in H^1$ and $\lambda < 0$ these solutions belong to H^3 [18])

$$-(\Delta + \lambda)U_f = F \qquad -(\Delta + \lambda)U_g = G_1$$

one finds that $U_r = U_f + (\alpha^2 + \lambda)u_1 U_g$. The condition (11) can be rewritten as

$$u_0 + \frac{\kappa_2}{\gamma_2} u_1 = \tilde{A}\left(\frac{\partial U_f(0,0)}{\partial x} + (\alpha^2 + \lambda)\frac{\partial U_g(0,0)}{\partial x}\, u_1 + 2Wu_1\right).$$

Together with the equation in the second component of the spectral problem this gives the system of linear equations for the constants u_0 and u_1. The determinant of this system is

$$\mathrm{Det} = 1 - (\beta^2 + \lambda)\left(\frac{\kappa_2}{\gamma_2} - \tilde{A}\left(2W + (\alpha^2 + \lambda^2)\frac{\partial U_g(0,0)}{\partial x}\right)\right).$$

For $\lambda = -\beta^2$ the determinant is equal to one and the system has a solution for any right-hand side.

That is, the deficiency index of $\mathcal{A}$ for $\lambda = -\beta^2$ is equal to zero. The operator is essentially self-adjoint and as its domain is closed one concludes that $\mathcal{A}^* = \mathcal{A}$. □

3.3. Parametrization of the zero-range potentials

Condition (11) is difficult to check and for the zero-range potentials for differential operator the restrictions are usually formulated for the coefficients of the local asymptotics of the function. The asymptotics, in the case of point dipole source, have the form

$$\mathcal{U}(x, y) \sim a\frac{x}{\pi r^2} + b + c\frac{x \log r}{2\pi} + \mathrm{d}x + \cdots \qquad r \to 0. \tag{12}$$

The component u_0 of the element $\mathcal{U}$ coincides with a in (12) and the component $U(x, y)$ has the asymptotics

$$U(x, y) \sim b + (c - \beta^2 a)\frac{x \log r}{2\pi} + (d - \chi_0 a)x + \cdots \qquad r \to 0.$$

Here one finds the regular part and the coefficient u_1

$$U_r(x, y) \sim b + (d - \chi_1 c - (\chi_0 - \beta^2\chi_1)a)x + \cdots \qquad u_1 = c - \beta^2 a.$$

Condition (11) can be rewritten for the coefficients a, b, c, d in the form

$$a - \kappa_2 c = A(d + Zc) \tag{13}$$

where A differs from $\tilde{A}$ in a real multiplier and

$$Z = \frac{\beta^2\kappa_2}{4\pi} + 2\frac{W}{\gamma_2} - \chi_1.$$

It is possible to choose the parameters of the scalar product in such a way that the formulae simplify. Namely $W = \kappa$ if

$$\kappa_2\alpha^2 = 1. \tag{14}$$

The quantity Z can be made equal to zero. For example, assuming that β is large, so that $\log(\beta/2) > \frac{3}{2} - C_E$ and taking

$$\gamma_2 = 1 \qquad \text{and} \qquad \kappa = \frac{1}{4\pi}\left(\log(\beta/2) + C_E - \frac{3}{2}\right) \tag{15}$$

yields $Z = 0$. Besides, if $\beta \to +\infty$, then $\kappa_2 \to +0$. Below, we assume that the parameters are taken as in (14) and (15) and in accordance with conditions (10). Still, that gives some arbitrariness in the choice of κ_2 and this will be discussed below. In that case the 'boundary' condition in the potential centre is formulated as

$$a + \kappa_2 c = Ad. \tag{16}$$

3.4. *Choice of the parameter*

The parameter A in (16) should be chosen so that the scattering pattern in the problem of scattering by the zero-range potential coincides in the highest order of $kH \to 0$ with the expression (3). The problem of scattering by the zero-range potential is formulated as follows. One considers the spectral problem

$$\mathcal{A}\mathcal{U} = k^2\mathcal{U}$$

for the functions $\mathcal{U}(x, y) = U(x, y) + u_0 G_0(x, y)$ that have the asymptotics (12) with the coefficients satisfying the condition (16) where A is fixed. It is easy to find that the solution of the problem of scattering by the zero-range potential has the form

$$U(x, y) = U^i(x, y) + U^r(x, y) + a\left(\frac{\mathrm{i}}{2}H_1^{(1)}(kr)k\cos\varphi - G_0(x, y)\right)$$
$$u_0 = a.$$

The amplitude a is chosen from condition (16). The coefficient d in the asymptotics (12) of the field $\mathcal{U}(x, y)$ depends on the incident field and on the amplitude a, by the formula

$$d = 2\mathrm{i}k\cos\varphi_0 - a\frac{k^2}{4\pi}(2\log(k/2) + 2C_E - 1 - \mathrm{i}\pi)$$

and

$$c = -ak^2.$$

Finally, one finds

$$a = \frac{2\mathrm{i}kA\cos\varphi_0}{1 - k^2\kappa_2 + k^2\frac{A}{4\pi}(2\log(k/2) + 2C_E - 1 - \mathrm{i}\pi)} \tag{17}$$

and

$$\begin{aligned}\Psi_0(\varphi,\varphi_0) &= -a\frac{k\cos\varphi}{2\pi} \\ &\approx -\frac{\mathrm{i}}{\pi}Ak^2\cos\varphi\cos\varphi_0. \end{aligned} \tag{18}$$

Here we assumed $\kappa_2 \ll 1$ and neglected the terms proportional to $k^2\kappa_2$ and k^2A. This allows the parameter A to be chosen independently of the incident wave, its wavenumber k and incidence angle φ_0:

$$A = -\frac{\pi}{2}H^2. \tag{19}$$

Formulae (17) and (18) are exact and the model of the zero-range potential leads to the field that exactly satisfies the reciprocity principle and the optical theorem [19] which follows from the theory of self-adjoint operators.

4. Conclusion

The constructed operator model of the scattering by a protruding stiffener simplifies the analysis of the scattering effects. In particular, it is easy to consider the periodic set of such stiffeners and look at the dynamics of the wave processes.

The main difference of this approach with respect to the scheme of [6–9] is in the use of a specific space where the operator is considered. The necessity to introduce δ'-sources forced the two-component space to be used. The derivative of the function from the domain of the operator should be defined in the potential centre and this requires the space H^1 to be chosen for the component $U(x, y)$. A different approach based on the use of spaces with appropriate weight is developed in [20]. However, an operator pencil is constructed there, that is the spectral parameter appears in the matrix of the condition similar to (18).

However, our approach leads to cumbersome constructions and its modification for the analysis of the scattering by elastic screens [21, 22] appears difficult due to the use of the space H^1 instead of the usual L_2. The final formula (16) allows the coefficient κ_2 to be taken equal to zero, however, this means that the other parameters should be taken as infinitely large. If the conditions (14) and (15) are assumed, then taking $\kappa_2 = 0$ yields $\beta^2 = +\infty$. In contrast, taking β as fixed and rejecting the conditions (15) forces κ to be taken as infinitely large for the norm to be positive (see (7)). Finally, if the positiveness of the norm is not required and κ is taken to be finite, the space $\mathcal{H}$ becomes, with indefinite metrics, similar to [12–14].

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